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Mixed-state entanglement and quantum teleportation through noisy channels

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Abstract

The quantum teleportation with a noisy EPR state is discussed. Using an optimal decomposition technique, we compute the concurrence, entanglement of formation and Groverian measure for various noisy EPR resources. It is shown analytically that all entanglement measures reduce to zero when $\bar{F} \leqslant 2/3$, where $\bar{F}$ is an average fidelity between Alice and Bob. This fact indicates that the entanglement is a genuine physical resource for the teleportation process. This fact gives valuable clues to the optimal decomposition for higher-qubit mixed states. As an example, the optimal decompositions for the 3-qubit mixed states are discussed by adopting a teleportation with a W-state.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Entanglement of quantum states plays a crucial role in modern quantum information theories [1]. Although we do not have a general theory of the quantum entanglement, many physicists believe that it is a physical resource which makes quantum computers outperform classical ones [2]. Thus in order to quantify the entanglement of a given quantum state many entanglement measures were constructed during the last decade. The basic entanglement measure is the entanglement of formation [3–6]. Generally, entanglement of formation is defined in any bipartite system. For a pure state if $|\psi\rangle$ is the state of the whole system, the entanglement of formation $\mathcal{E}(\psi)$ is defined as the von Neumann entropy $\mathcal{E}(\psi) = -\text{Tr}\rho \log_2 \rho$, where ρ is the partial trace over either of the two subsystems. Another measure we would like to use in this paper is Groverian measure [7]. The Groverian measure $G(\psi)$ for the given n-qubit quantum state $|\psi\rangle$ is defined using a quantity

$$P_{\max}(\psi) = \max_{|q_1\rangle,\ldots,|q_n\rangle} |\langle q_1| \ldots \langle q_n|\psi\rangle|^2 \qquad (1.1)$$

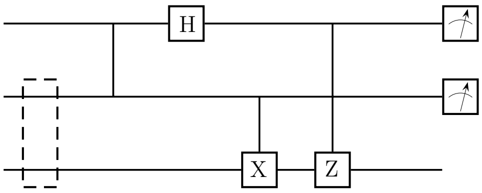

Figure 1. A quantum circuit for quantum teleportation through noisy channels with EPR state. The top two lines belong to Alice while the bottom line belongs to Bob. The dotted box represents noisy channels, which makes the EPR state a mixed state.

where $|q_i\rangle$'s are single-qubit states. In fact, $P_{\max}(\psi)$ is the maximal probability of success in the Grover's search algorithm [8] when $|\psi\rangle$ is used as an initial state. Roughly speaking, $P_{\max}$ quantifies a distance between a given n-qubit state $|\psi\rangle$ and a set of product states. Therefore, the entanglement should decrease with increasing $P_{\max}$. For this reason the Groverian measure is defined as $G(\psi) = \sqrt{1 - P_{\max}(\psi)}$. For 2-qubit pure states $P_{\max}$ can be analytically computed [9], whose expression is

$$P_{\max} = \tfrac{1}{2}[1 + \sqrt{1 - 4 \det \rho}] \tag{1.2}$$

where ρ is the partial trace over either of the 2-qubits. Recently, $P_{\max}$ for some 3-qubit states were also computed analytically [10–12] by exploiting a theorem of [13]. Although much progress has been made recently for understanding the general features of pure-state entanglement, it seems to be far from complete understanding.

The purpose of this paper is to examine the physical role of mixed-state entanglement. In order to address this issue it is convenient to consider the quantum teleportation [14] when the quantum channel is affected by noise. The effect of noise in teleportation was discussed in [15]. We would like to summarize the paper briefly in the following. Let us consider the usual situation of the teleportation: Alice and Bob share an EPR channel

$$|\beta_{00}\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{1.3}$$

and Alice wants to send a single-qubit state

$$|\psi_{\text{in}}\rangle = \cos\left(\frac{\theta}{2}\right) e^{i\phi/2}|0\rangle + \sin\left(\frac{\theta}{2}\right) e^{-i\phi/2}|1\rangle \tag{1.4}$$

to Bob. We assume, however, that the perfect EPR state was not prepared initially due to noise. In terms of density operator language this means that instead of $\rho_{\text{EPR}} = |\beta_{00}\rangle\langle\beta_{00}|$ the imperfect density operator $\varepsilon(\rho_{\text{EPR}})$ was made initially, where ε is a quantum operation. Since $\varepsilon(\rho_{\text{EPR}}) \neq \rho_{\text{EPR}}$ generally, Alice cannot send $|\psi_{\text{in}}\rangle$ perfectly to her remote recipient. This situation is depicted in figure 1. In this figure the top two lines belong to Alice while the bottom line belongs to Bob. The density operator ρ_{in} is $|\psi_{\text{in}}\rangle\langle\psi_{\text{in}}|$ and ρ_{out} is a state Bob receieves from Alice. The dotted box represents an imperfect EPR resource produced initially due to the noise.

Two questions naturally arise at this stage. The first one is what the explicit expression of $\varepsilon(\rho_{\text{EPR}})$ is. The second one is how much information Alice can send to Bob. Obviously, the answers are dependent on what type of noise we take into account. To address the first question authors in [15] used a master equation in the Lindbald form [16]

$$\frac{\partial \sigma}{\partial t} = -i[H_S, \sigma] + \sum_{i,\alpha} \left(L_{i,\alpha} \sigma L_{i,\alpha}^{\dagger} - \frac{1}{2}\left\{ L_{i,\alpha}^{\dagger} L_{i,\alpha}, \sigma \right\} \right) \tag{1.5}$$

where $\sigma \equiv \varepsilon(\rho_{\rm EPR})$ and $L_{i,\alpha}$ is an Lindbald operator which represents the type of noise. In order to simplify the situation simple types of noise $L_{i,\alpha} \equiv \sqrt{\kappa}\sigma_\alpha^{(i)}$ have been chosen in [15] which acts on the ith qubit to describe decoherence, where $\sigma_\alpha^{(i)}$ denotes the Pauli matrix of the ith qubit with $\alpha = x, y, z$. The constant κ is approximately equals to the inverse of decoherence time. The master equation approach is shown to be equivalent to the usual quantum operation approach for the description of noise in an open quantum system [1]. Solving a master equation (1.5), we can now derive $\varepsilon(\rho_{\rm EPR})$ explicitly. If we choose noises with same direction, i.e. $(L_{2,x}, L_{3,x})$, $(L_{2,y}, L_{3,y})$ or $(L_{2,z}, L_{3,z})$, equation (1.5) provides

$$\varepsilon_x(\rho_{\rm EPR}) = \frac{1}{2}\begin{pmatrix} \tau_+ & 0 & 0 & \tau_+ \\ 0 & \tau_- & \tau_- & 0 \\ 0 & \tau_- & \tau_- & 0 \\ \tau_+ & 0 & 0 & \tau_+ \end{pmatrix}, \qquad \varepsilon_y(\rho_{\rm EPR}) = \frac{1}{2}\begin{pmatrix} \tau_+ & 0 & 0 & \tau_+ \\ 0 & \tau_- & -\tau_- & 0 \\ 0 & -\tau_- & \tau_- & 0 \\ \tau_+ & 0 & 0 & \tau_+ \end{pmatrix}$$

$$\varepsilon_z(\rho_{\rm EPR}) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & {\rm e}^{-4\kappa t} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ {\rm e}^{-4\kappa t} & 0 & 0 & 1 \end{pmatrix}, \tag{1.6}$$

where $\tau_\pm = (1 \pm {\rm e}^{-4\kappa t})/2$. If one chooses the isotropic noise, equation (1.5) yields

$$\varepsilon_I(\rho_{\rm EPR}) = \frac{1}{2}\begin{pmatrix} \tilde\tau_+ & 0 & 0 & 2\tilde\tau_+ - 1 \\ 0 & \tilde\tau_- & 0 & 0 \\ 0 & 0 & \tilde\tau_- & 0 \\ 2\tilde\tau_+ - 1 & 0 & 0 & \tilde\tau_+ \end{pmatrix} \tag{1.7}$$

where $\tilde\tau_\pm = (1 \pm {\rm e}^{-8\kappa t})/2$.

To address the second issue we consider a square of fidelity between $\rho_{\rm in}$ and $\rho_{\rm out}$

$$F(\rho_{\rm in}, \rho_{\rm out}) = \langle\psi_{\rm in}|\rho_{\rm out}|\psi_{\rm in}\rangle \equiv F(\theta, \phi). \tag{1.8}$$

Then how much information Alice can send to Bob with imperfect EPR resource $\varepsilon(\rho_{\rm EPR})$ can be measured by the average fidelity

$$\bar F \equiv \frac{1}{4\pi}\int_0^{2\pi} {\rm d}\phi \int_0^\pi {\rm d}\theta \sin\theta F(\theta, \phi). \tag{1.9}$$

Thus the perfect teleportation means $\bar F = 1$. Reference [15] has shown that for the same-axis noises the average fidelities become

$$\bar F_x = \bar F_y = \bar F_z = \tfrac{2}{3} + \tfrac{1}{3}{\rm e}^{-4\kappa t} \tag{1.10}$$

while for the case of the isotropic noise $\bar F$ becomes

$$\bar F_I = \tfrac{1}{2} + \tfrac{1}{2}{\rm e}^{-8\kappa t}. \tag{1.11}$$

Regardless of types of the noisy channels $\bar F$ decays as κt increases.

What kind of information on the average fidelity $\bar F$ can be obtained from the entanglement of the mixed states $\varepsilon_\alpha(\rho_{\rm EPR})$ $(\alpha = x, y, z)$ and $\varepsilon_I(\rho_{\rm EPR})$ or vice versa? To address this question is the main motivation of this paper. Since $\bar F$ decreases with increasing κt, we can conjecture that the effect of noises generally disentangles the mixed states provided the entanglement is a genuine resource for the teleportation. Since, furthermore, $\bar F = 2/3$ corresponds to the best possible score when Alice and Bob communicate with each other through the classical channel [17], this fact implies that $\varepsilon(\rho_{\rm EPR})$ does not play any role as an entanglement resource when $\bar F \leqslant 2/3$. Thus we can conjecture that $\varepsilon_\alpha(\rho_{\rm EPR})$ $(\alpha = x, y, z)$ should be separable states as κt approaches infinity while $\varepsilon_I(\rho_{\rm EPR})$ becomes separable when $\kappa t \geqslant \mu_* = (1/8)\ln 3$. If our conjecture is right, we can conjecture $\bar{F}$ from the entanglement of the mixed-state resource without any calculation. Reversely, we can conjecture the entanglement of mixed states from the average fidelity. This means that entanglement is a genuine resource in the teleportation process even if noises are involved. Since an explicit calculation of the n-qubit mixed-state entanglement is highly non-trivial when $n \geqslant 3$,[4] it may give a valuable tool for the approximate conjecture of the entanglement.

We will show that the above-mentioned conjectures on the relation between entanglement of mixed-state and $\bar{F}$ are perfectly correct. This paper is organized as follows. In section 2 we discuss the entanglement measures for the mixed states and their inter-relations. It is found that not only the entanglement of formation but also the Groverian measure are monotonically related to the concurrence. This fact indicates that the optimal ensemble for the concurrence is also optimal for the Groverian measure. In section 3 we compute explicitly the concurrence, entanglement of formation and Groverian measure for various mixed-states obtained by same-axis and isotropic noises. The results of the computation are compared to the average fidelity $\bar{F}$. It is shown that as we conjectured, all entanglement measures become zero when $\bar{F} \leqslant 2/3$. To confirm that our conjecture is right, we also compute the entanglement measures and average fidelity for different-axis noises in section 4. In these cases the results perfectly agree with our conjecture. In section 5 the optimal decomposition for the higher-qubit mixed states is discussed. Especially, the case of 3-qubit mixed-state is discussed by adopting quantum teleportation with W-state. Also the calculability for the second definition of the Groverian measure is briefly discussed in the same section.

2. Entanglement of mixed states

There are many measures which quantify the entanglement of the mixed states. Among them we will use in this paper the entanglement of formation and the Groverian measure.

As we said in the previous section, the entanglement of formation for any pure bipartite system is defined as a von Neumann entropy of its subsystems. Then using a convex roof construction [18, 19], one can extend the definition of the entanglement of formation to the full state space in a natural way as

$$\mathcal{E}(\rho) = \min \sum_j P_j \mathcal{E}(\rho_j) \tag{2.1}$$

where minimum is taken over all possible ensembles of pure states ρ_j with $0 \leqslant P_j \leqslant 1$. In [4, 5] it was shown how to construct the optimal ensemble, where the minimization in equation (2.1) is naturally taken in the 2-qubit system.

A convex roof method can also be used to extend the definition of the Groverian measure in the full state space

$$G(\rho) = \min \sum_j P_j G(\rho_j) \tag{2.2}$$

where minimum is taken over all possible ensembles of pure states. Since the Groverian measure for the pure state is an entanglement monotone [20], it is not difficult to prove that $G(\rho)$ in equation (2.2) is also monotone even if ρ is a mixed state.

However, there is different extension of the Groverian measure from the aspect of the operational treatment of the entanglement [21]. In [21], the Groverian measure for the mixed state is defined as

$$\tilde{G}(\rho) = \sqrt{1 - \max_{\sigma \in \mathcal{S}} F^2(\rho, \sigma)}, \tag{2.3}$$

[4] For some entanglement measures it is also highly non-trivial to compute it even for $n = 2$.

where $\mathcal{S}$ is a set of separable states and $F(\rho, \sigma)$ is a fidelity defined $F(\rho, \sigma) = \mathrm{Tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}$. It was shown in [21] that $\tilde{G}(\rho)$ is also an entanglement monotone. Following Uhlmann theorem [22] one can re-express $\tilde{G}(\rho)$ in a form

$$\tilde{G}(\rho) = \sqrt{1 - \max_{|\phi\rangle} \max_{|\psi\rangle} |\langle\phi|\psi\rangle|^2} \tag{2.4}$$

where $|\phi\rangle$ and $|\psi\rangle$ are purifications of σ and ρ respectively[5].

Now, we would like to comment on how the optimization for the Groverian measure defined in equation (2.2) is taken. In order to describe this it is convenient to comment first on how the optimization for the entanglement of formation was taken in [4, 5]. First, the authors in these references notified that in the pure 2-qubit state $|\psi\rangle$ the entanglement of formation $\mathcal{E}(\psi)$ and concurrence $\mathcal{C}(\psi)$ are related to each other in a form

$$\mathcal{E}(\psi) = h\left(\frac{1 + \sqrt{1 - \mathcal{C}^2(\psi)}}{2}\right) \tag{2.5}$$

where $h(x) \equiv -x\log_2 x - (1-x)\log_2(1-x)$. Thus $\mathcal{E}(\mathcal{C})$ is monotonically increasing from 0 to 1 as $\mathcal{C}$ goes from 0 to 1. For the mixed states, therefore, optimization for the concurrence in all possible pure-state ensembles naturally coincides with optimization for the entanglement of formation. Second, the authors in [4] found the optimization for the concurrence by making use of some geometrical argument when the density matrix has two or three zero eigenvalues. Finally, Wootters derived the optimal ensemble for arbitrary 2-qubit mixed states in [5]. We should note that the Groverian measure for the arbitrary 2-qubit pure state $|\psi\rangle$ is related to the concurrence in a form

$$G(\psi) = \tfrac{1}{\sqrt{2}}\left(1 - \sqrt{1 - \mathcal{C}^2(\psi)}\right)^{1/2}. \tag{2.6}$$

Like the entanglement of formation, therefore, $G(\mathcal{C})$ is also a monotonic function from 0 to $1/\sqrt{2}$ as $\mathcal{C}$ goes from 0 to 1. This supports that the optimization for the concurrence in all possible ensembles of pure states coincides with not only that for the entanglement of formation but also that for the Groverian measure defined in equation (2.2).

Although, therefore, the optimization for the first Groverian measure $G(\rho)$ is possible, the optimization for the second Groverian measure $\tilde{G}(\rho)$ seems to be highly non-trivial because it is defined by 4-qubit pure states via the purification and the Uhlmann theorem. In this paper we will use $\mathcal{E}(\rho)$ and $G(\rho)$ to confirm our conjecture on the relation between the mixed-state entanglement and the average fidelity $\bar{F}$.

3. Same-axis and isotropic noises

In this section we would like to compute the entanglement for the mixed states given in equations (1.6) and (1.7). Before starting computation it is convenient for later use to introduce a 'magic basis' [18]:

$$\begin{aligned} |e_1\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad & |e_2\rangle &= \frac{\mathrm{i}}{\sqrt{2}}(|00\rangle - |11\rangle) \\ |e_3\rangle &= \frac{\mathrm{i}}{\sqrt{2}}(|01\rangle + |10\rangle) \qquad & |e_4\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \end{aligned} \tag{3.1}$$

[5] In fact, one can remove the optimization on $|\psi\rangle$ [1], which yields

$$\tilde{G}(\rho) = \sqrt{1 - \max_{|\phi\rangle} |\langle\phi|\psi\rangle|^2}.$$

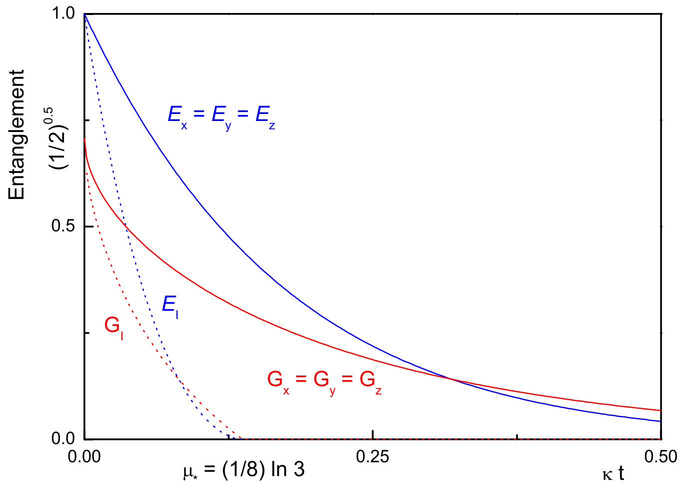

Figure 2. The κt-dependence of the entanglement formation and Groverian measure for $\varepsilon_\alpha(\rho_{\text{EPR}})(\alpha = x, y, z)$ and $\varepsilon_I(\rho_{\text{EPR}})$. Regardless of noise types the entanglement decreases with increasing κt. This means that the noises generally disentangle the quantum channel. For the isotropic noisy channel $\mathcal{E}_I$ and G_I become zero when $\kappa t \geqslant \mu_* = (1/8)\ln 3$, where the average fidelity $\bar{F}$ is less than $2/3$.

Now let us consider $(L_{2,x}, L_{3,x})$ noise which makes the EPR resource as $\varepsilon_x(\rho_{\text{EPR}})$ in equation (1.6). Since $\varepsilon_x(\rho_{\text{EPR}})$ has two zero eigenvalues, one can construct the optimal ensemble of pure states by two different ways explained in [4] and [5] respectively. It is not difficult to show that both methods yield the same optimal ensemble whose explicit expression is

$$\varepsilon_x(\rho_{\text{EPR}}) = \sum_{i=1}^{2} P_i |X_i\rangle\langle X_i| \tag{3.2}$$

where $P_1 = P_2 = 1/2$ and

$$|X_1\rangle = \sqrt{\tau_+}|e_1\rangle + \mathrm{i}\sqrt{\tau_-}|e_3\rangle \qquad |X_2\rangle = \sqrt{\tau_+}|e_1\rangle - \mathrm{i}\sqrt{\tau_-}|e_3\rangle. \tag{3.3}$$

Since the concurrence for the arbitrary 2-qubit state $|\psi\rangle = \sum_{i=1}^{4} \alpha_i|e_i\rangle$ is $|\sum_i \alpha_i^2|$, $|X_1\rangle$ and $|X_2\rangle$ have the same concurrence

$$\mathcal{C}_x = \mathcal{C}(|X_1\rangle) = \mathcal{C}(|X_2\rangle) = \tau_+ - \tau_- = \mathrm{e}^{-4\kappa t}. \tag{3.4}$$

Thus the entanglement of formation $\mathcal{E}_x$ and the Groverian measure G_x can easily be computed by equations (2.5) and (2.6), respectively. The κt-dependences of $\mathcal{E}_x$ and G_x are plotted in figure 2 as solid lines. As expected, $\mathcal{E}_x$ and G_x decrease from 1 and $1/\sqrt{2}$ to 0 as κt goes from 0 to ∞. This means that the noise disentangles $\varepsilon_x(\rho_{\text{EPR}})$ as we conjectured. Since $\mathcal{E}_x = G_x = 0$ at the $\kappa t \to \infty$, $\varepsilon_x(\rho_{\text{EPR}})$ should be separable in this limit. We can confirm this directly from equation (3.3) because $|X_1\rangle$ and $|X_2\rangle$ reduce to $(|0\rangle \mp |1\rangle)/\sqrt{2} \otimes (|0\rangle \mp |1\rangle)/\sqrt{2}$ at the $\kappa t \to \infty$ limit. If one constructs the optimal ensembles for $\varepsilon_y(\rho_{\text{EPR}})$ and $\varepsilon_z(\rho_{\text{EPR}})$, one can show in the same way that $\varepsilon_y(\rho_{\text{EPR}}) = \sum_{i=1}^{2} P_i|Y_i\rangle\langle Y_i|$ where $P_1 = P_2 = 1/2$ and

$$|Y_1\rangle = \sqrt{\tau_+}|e_1\rangle + \mathrm{i}\sqrt{\tau_-}|e_4\rangle \qquad |Y_2\rangle = \sqrt{\tau_+}|e_1\rangle - \mathrm{i}\sqrt{\tau_-}|e_4\rangle, \tag{3.5}$$

and $\varepsilon_z(\rho_{\text{EPR}}) = \sum_{i=1}^{2} P_i|Z_i\rangle\langle Z_i|$ where $P_1 = P_2 = 1/2$ and

$$|Z_1\rangle = \sqrt{\tau_+}|e_1\rangle + \mathrm{i}\sqrt{\tau_-}|e_2\rangle \qquad |Z_2\rangle = \sqrt{\tau_+}|e_1\rangle - \mathrm{i}\sqrt{\tau_-}|e_2\rangle. \tag{3.6}$$

It is easy to show $\mathcal{E}_x = \mathcal{E}_y = \mathcal{E}_z$ and $G_x = G_y = G_z$.

Now, let us consider $\varepsilon_I(\rho_{\text{EPR}})$. Taking into account the partial transposition [23–25] of $\varepsilon_I(\rho_{\text{EPR}})$ with respect to its subsystems, one can realize that $\varepsilon_I(\rho_{\text{EPR}})$ is separable when $\kappa t \geqslant \mu_* = (1/8)\ln 3$. Following [5], one can derive the separable decomposition $\varepsilon_I(\rho_{\text{EPR}}) = \sum_{i=1}^4 |S_i\rangle\langle S_i|$ in this region, where $|S_i\rangle$ are un-normalized vectors defined as

$$\begin{aligned}
|S_1\rangle &= \tfrac{1}{2}(\mathrm{e}^{\mathrm{i}\theta_1}|x_1\rangle + \mathrm{e}^{\mathrm{i}\theta_2}|x_2\rangle + \mathrm{e}^{\mathrm{i}\theta_3}|x_3\rangle + \mathrm{e}^{\mathrm{i}\theta_4}|x_4\rangle)\\
|S_2\rangle &= \tfrac{1}{2}(\mathrm{e}^{\mathrm{i}\theta_1}|x_1\rangle + \mathrm{e}^{\mathrm{i}\theta_2}|x_2\rangle - \mathrm{e}^{\mathrm{i}\theta_3}|x_3\rangle - \mathrm{e}^{\mathrm{i}\theta_4}|x_4\rangle)\\
|S_1\rangle &= \tfrac{1}{2}(\mathrm{e}^{\mathrm{i}\theta_1}|x_1\rangle - \mathrm{e}^{\mathrm{i}\theta_2}|x_2\rangle + \mathrm{e}^{\mathrm{i}\theta_3}|x_3\rangle - \mathrm{e}^{\mathrm{i}\theta_4}|x_4\rangle)\\
|S_1\rangle &= \tfrac{1}{2}(\mathrm{e}^{\mathrm{i}\theta_1}|x_1\rangle - \mathrm{e}^{\mathrm{i}\theta_2}|x_2\rangle - \mathrm{e}^{\mathrm{i}\theta_3}|x_3\rangle + \mathrm{e}^{\mathrm{i}\theta_4}|x_4\rangle).
\end{aligned} \tag{3.7}$$

In equation (3.7) $|x_i\rangle$ are

$$\begin{aligned}
|x_1\rangle &= -\mathrm{i}\sqrt{\frac{3\tilde{\tau}_+ - 1}{2}}|e_1\rangle \qquad |x_2\rangle = -\mathrm{i}\sqrt{\frac{\tilde{\tau}_-}{2}}|e_2\rangle\\
|x_3\rangle &= -\mathrm{i}\sqrt{\frac{\tilde{\tau}_-}{2}}|e_3\rangle \qquad |x_4\rangle = -\mathrm{i}\sqrt{\frac{\tilde{\tau}_-}{2}}|e_4\rangle
\end{aligned} \tag{3.8}$$

and θ_i's satisfy

$$\frac{3\tilde{\tau}_+ - 1}{\tilde{\tau}_-}\mathrm{e}^{2\mathrm{i}\theta_1} + (\mathrm{e}^{2\mathrm{i}\theta_2} + \mathrm{e}^{2\mathrm{i}\theta_3} + \mathrm{e}^{2\mathrm{i}\theta_4}) = 0. \tag{3.9}$$

Since all $|S_i\rangle$ have zero concurrence provided equation (3.9) holds, $\varepsilon_I(\rho_{\text{EPR}})$ becomes separable in the region $\kappa t \geqslant \mu_*$. In order to see this explicitly let us consider the boundary of this region $\kappa t = \mu_*$. At this point we have $\theta_1 = 0$ and $\theta_2 = \theta_3 = \theta_4 = \pi/2$ which yield the following separable decomposition $\varepsilon_I(\rho_{\text{EPR}}) = \sum_{i=1}^4 P_i|\tilde{s}_i\rangle\langle\tilde{s}_i|$ where $P_1 = P_2 = P_3 = P_4 = 1/4$ and

$$\begin{aligned}
|\tilde{s}_1\rangle &= (\omega_-|0\rangle - \omega_+\mathrm{e}^{\mathrm{i}\pi/4}|1\rangle) \otimes (\omega_-|0\rangle - \omega_+\mathrm{e}^{-\mathrm{i}\pi/4}|1\rangle)\\
|\tilde{s}_2\rangle &= (\omega_-|0\rangle + \omega_+\mathrm{e}^{\mathrm{i}\pi/4}|1\rangle) \otimes (\omega_-|0\rangle + \omega_+\mathrm{e}^{-\mathrm{i}\pi/4}|1\rangle)\\
|\tilde{s}_3\rangle &= (\omega_+|0\rangle - \omega_-\mathrm{e}^{-\mathrm{i}\pi/4}|1\rangle) \otimes (\omega_+|0\rangle - \omega_-\mathrm{e}^{\mathrm{i}\pi/4}|1\rangle)\\
|\tilde{s}_4\rangle &= (\omega_+|0\rangle + \omega_-\mathrm{e}^{-\mathrm{i}\pi/4}|1\rangle) \otimes (\omega_+|0\rangle + \omega_-\mathrm{e}^{\mathrm{i}\pi/4}|1\rangle)
\end{aligned} \tag{3.10}$$

with $\omega_\pm = (\sqrt{3}(\sqrt{3} \pm 1)/6)^{1/2}$.

In the $\kappa t \leqslant \mu_*$ region $\varepsilon_I(\rho_{\text{EPR}})$ is generally entangled. The optimal ensemble of pure states can be constructed following [5]. The final expression of decomposition is $\varepsilon_I(\rho_{\text{EPR}}) = \sum_{i=1}^4 P_i|I_i\rangle\langle I_i|$ where $P_1 = P_2 = P_3 = P_4 = 1/4$ and

$$\begin{aligned}
|I_1\rangle &= \sqrt{\lambda_1}|e_1\rangle - \mathrm{i}\sqrt{3\lambda_2}|e_2\rangle\\
|I_2\rangle &= \sqrt{\lambda_1}|e_1\rangle + \mathrm{i}\sqrt{\frac{\lambda_2}{3}}|e_2\rangle - 2\mathrm{i}\sqrt{\frac{2\lambda_2}{3}}|e_3\rangle\\
|I_3\rangle &= \sqrt{\lambda_1}|e_1\rangle + \mathrm{i}\sqrt{\frac{\lambda_2}{3}}|e_2\rangle + \mathrm{i}\sqrt{\frac{2\lambda_2}{3}}|e_3\rangle - \mathrm{i}\sqrt{2\lambda_2}|e_4\rangle\\
|I_4\rangle &= \sqrt{\lambda_1}|e_1\rangle + \mathrm{i}\sqrt{\frac{\lambda_2}{3}}|e_2\rangle + \mathrm{i}\sqrt{\frac{2\lambda_2}{3}}|e_3\rangle + \mathrm{i}\sqrt{2\lambda_2}|e_4\rangle
\end{aligned} \tag{3.11}$$

where $\lambda_1 = (3\tilde{\tau}_+ - 1)/2$ and $\lambda_2 = \tilde{\tau}_-/2$. It is easy to show that in the region $\kappa t \leqslant \mu_* \varepsilon_I(\rho_{\text{EPR}})$ has a concurrence

$$\mathcal{C}_I = \lambda_1 - 3\lambda_2 = \tfrac{3}{2}\left(\mathrm{e}^{-8\kappa t} - \tfrac{1}{3}\right). \tag{3.12}$$

Since $\varepsilon_I(\rho_{\text{EPR}})$ is a separable mixed state at $\kappa t \geqslant \mu_*$, $\mathcal{C}_I$ equals zero in this region. Thus we can write it in a form

$$\mathcal{C}_I = \text{Max}(\lambda_1 - 3\lambda_2, 0). \tag{3.13}$$

Inserting equation (3.13) into equations (2.5) and (2.6), one can easily compute the entanglement of formation $\mathcal{E}_I$ and the Groverian measure G_I for $\varepsilon_I(\rho_{\mathrm{EPR}})$.

The κt-dependences of $\mathcal{E}_I$ and G_I are plotted in figure 2 as dotted lines. As we conjectured in section 1, $\mathcal{E}_I$ and G_I decrease from 1 and $1/\sqrt{2}$ to 0 as κt goes from 0 to μ_*. This means that when $\bar{F} \leqslant 2/3$, $\varepsilon_I(\rho_{\mathrm{EPR}})$ cannot play any role as a quantum channel. This fact also indicates that the entanglement is a genuine resource for the quantum communication. In order to confirm that our conjecture is right, we will consider the different-axis noises in the following section.

4. Different-axis noises

In this section we would like to consider the different-axis noises to confirm that our conjecture is right. First let us consider $(L_{2,x}, L_{3,z})$ noise. For this case the master equation (1.5) changes the EPR state ρ_{EPR} into

$$\varepsilon_{xz}(\rho_{\mathrm{EPR}}) = \frac{1}{2}\begin{pmatrix} \nu_+ & 0 & 0 & \mathrm{e}^{-2\kappa t}\nu_+ \\ 0 & \nu_- & \mathrm{e}^{-2\kappa t}\nu_- & 0 \\ 0 & \mathrm{e}^{-2\kappa t}\nu_- & \nu_- & 0 \\ \mathrm{e}^{-2\kappa t}\nu_+ & 0 & 0 & \nu_+ \end{pmatrix} \tag{4.1}$$

where $\nu_\pm = (1 \pm \mathrm{e}^{-2\kappa t})/2$. Following the calculation of [15], one can easily show that the average fidelity in this noise channel becomes

$$\bar{F} = \tfrac{1}{6}(3 + 2\,\mathrm{e}^{-2\kappa t} + \mathrm{e}^{-4\kappa t}). \tag{4.2}$$

Thus $\bar{F}$ becomes less than 2/3 when $\kappa t \geqslant \nu_* = \ln(1+\sqrt{2})/2$. We expect that $\varepsilon_{xz}(\rho_{\mathrm{EPR}})$ becomes separable in the region $\kappa t \geqslant \nu_*$. In fact, in this region $\varepsilon_{xz}(\rho_{\mathrm{EPR}})$ can be expressed as $\varepsilon_{xz}(\rho_{\mathrm{EPR}}) = \sum_{i=1}^{4} |\bar{s}_i\rangle\langle\bar{s}_i|$ where $|\bar{s}_i\rangle$ are unnormalized vectors defined by the same as equation (3.7), but $|x_i\rangle$ are

$$\begin{aligned} |x_1\rangle &= -\mathrm{i}\nu_+|e_1\rangle & |x_2\rangle &= -\mathrm{i}\sqrt{\nu_+\nu_-}|e_2\rangle \\ |x_3\rangle &= -\mathrm{i}\sqrt{\nu_+\nu_-}|e_3\rangle & |x_4\rangle &= -\mathrm{i}\nu_-|e_4\rangle \end{aligned} \tag{4.3}$$

and θ_i's satisfy

$$\mathrm{e}^{2\mathrm{i}\theta_1}\nu_+^2 + (\mathrm{e}^{2\mathrm{i}\theta_2} + \mathrm{e}^{2\mathrm{i}\theta_3})\nu_+\nu_- + \mathrm{e}^{2\mathrm{i}\theta_4}\nu_-^2 = 0. \tag{4.4}$$

Since all $|\bar{s}_i\rangle$ have zero concurrence, $\varepsilon_{xz}(\rho_{\mathrm{EPR}})$ is manifestly separable in $\kappa t \geqslant \nu_*$ as expected.

In the region $\kappa t \leqslant \nu_*$ we can derive an optimal ensemble of pure states. It needs a tedious calculation, and the final expression is $\varepsilon_{xz}(\rho_{\mathrm{EPR}}) = \sum_{i=1}^{4} P_i|XZ_i\rangle\langle XZ_i|$ where $P_1 = P_2 = \nu_+/(1+2\nu_+)$, $P_3 = P_4 = 1/(2(1+2\nu_+))$ and

$$\begin{aligned} |XZ_1\rangle &= \nu_+|e_1\rangle - \mathrm{i}\sqrt{\nu_-(1+\nu_+)}|e_2\rangle \\ |XZ_2\rangle &= \nu_+|e_1\rangle + \mathrm{i}\nu_+\sqrt{\frac{\nu_-}{1+\nu_+}}|e_2\rangle - \mathrm{i}\nu_+\sqrt{\frac{\nu_-(1+2\nu_+)}{1+\nu_+}}|e_3\rangle \\ |XZ_3\rangle &= \nu_+|e_1\rangle + \mathrm{i}\nu_+\sqrt{\frac{\nu_-}{1+\nu_+}}|e_2\rangle + \mathrm{i}\nu_+\sqrt{\frac{\nu_-(1+2\nu_+)}{1+\nu_+}}|e_3\rangle - \mathrm{i}\nu_-\sqrt{1+2\nu_+}|e_4\rangle \\ |XZ_4\rangle &= \nu_+|e_1\rangle + \mathrm{i}\nu_+\sqrt{\frac{\nu_-}{1+\nu_+}}|e_2\rangle + \mathrm{i}\nu_+\sqrt{\frac{\nu_-(1+2\nu_+)}{1+\nu_+}}|e_3\rangle + \mathrm{i}\nu_-\sqrt{1+2\nu_+}|e_4\rangle. \end{aligned} \tag{4.5}$$

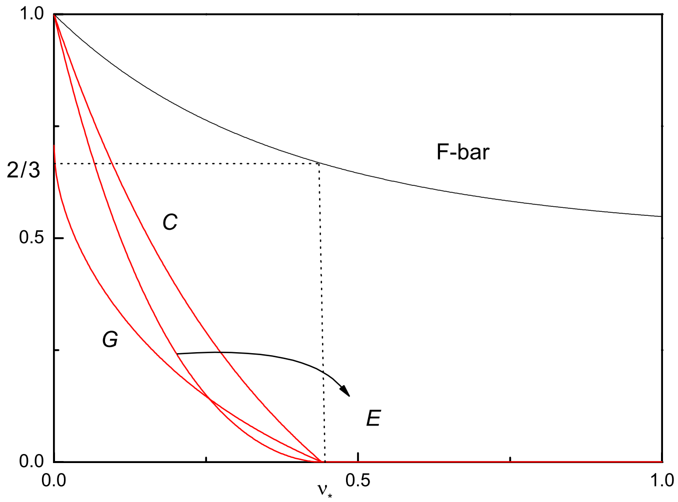

Figure 3. The κt-dependence of the average fidelity $\bar{F}$, entanglement of formation $\mathcal{E}$, concurrence $\mathcal{C}$ and Groverian measure G for different-axis noisy channels. As expected, all entanglement measures reduce to zero when $\kappa t \geqslant \nu_* = \ln(1+\sqrt{2})/2$.

Using equation (4.5) it is easy to compute the concurrence whose explicit expression is

$$\mathcal{C}_{xz} = \tfrac{1}{2}(\mathrm{e}^{-4\kappa t} + 2\,\mathrm{e}^{-2\kappa t} - 1) \tag{4.6}$$

at $\kappa t \leqslant \nu_*$. Thus in the full range of κt $\mathcal{C}(\rho_{\mathrm{EPR}})$ can be written as

$$\mathcal{C}_{xz} = \mathrm{Max}\left[\tfrac{1}{2}(\mathrm{e}^{-4\kappa t} + 2\,\mathrm{e}^{-2\kappa t} - 1), 0\right]. \tag{4.7}$$

Inserting equation (4.7) into equations (2.5) and (2.6), one can compute straightforwardly the entanglement of formation and the Groverian measure for $\varepsilon_{xz}(\rho_{\mathrm{EPR}})$.

For $(L_{2,x}, L_{3,y})$ and $(L_{2,y}, L_{3,z})$ noises the EPR state becomes respectively

$$\begin{aligned}
\varepsilon_{xy}(\rho_{\mathrm{EPR}}) &= \frac{1}{2}\begin{pmatrix} \tau_+ & 0 & 0 & \mathrm{e}^{-2\kappa t} \\ 0 & \tau_- & 0 & 0 \\ 0 & 0 & \tau_- & 0 \\ \mathrm{e}^{-2\kappa t} & 0 & 0 & \tau_+ \end{pmatrix} \\
\varepsilon_{yz}(\rho_{\mathrm{EPR}}) &= \frac{1}{2}\begin{pmatrix} \nu_+ & 0 & 0 & \mathrm{e}^{-2\kappa t}\nu_+ \\ 0 & \nu_- & \mathrm{e}^{-2\kappa t}\nu_- & 0 \\ 0 & \mathrm{e}^{-2\kappa t}\nu_- & \nu_- & 0 \\ \mathrm{e}^{-2\kappa t}\nu_+ & 0 & 0 & \nu_+ \end{pmatrix}.
\end{aligned} \tag{4.8}$$

It is not difficult to show that the average fidelity for these is equal to equation (4.2) and their concurrences are the same as equation (4.7), i.e. concurrence for $\varepsilon_{xz}(\rho_{\mathrm{EPR}})$. The optimal ensembles are $\varepsilon_{xy}(\rho_{\mathrm{EPR}}) = \sum_{i=1}^{4} P_i|XY_i\rangle\langle XY_i|$ and $\varepsilon_{yz}(\rho_{\mathrm{EPR}}) = \sum_{i=1}^{4} P_i|YZ_i\rangle\langle YZ_i|$, where $P_1 = P_2 = \nu_+/(1+2\nu_+)$ and $P_3 = P_4 = 1/(2(1+2\nu_+))$. The optimal pure states $|YZ_i\rangle$ can be obtained from $|XZ_i\rangle$ by interchanging $|e_3\rangle$ and $|e_4\rangle$. The optimal vectors $|XY_i\rangle$ are obtained from $|XZ_i\rangle$ by cyclic change, i.e. $|e_2\rangle \to |e_3\rangle, |e_3\rangle \to |e_4\rangle, |e_4\rangle \to |e_2\rangle$. The remaining different-axis noises $(L_{2,z}, L_{3,x})$, $(L_{2,z}, L_{3,y})$, $(L_{2,y}, L_{3,x})$ generate similar quantum channels to equations (4.1) and (4.8). They also yield the same average fidelities and the same concurrences.

The average fidelity $\bar{F}$, concurrence $\mathcal{C}$, entanglement of formation $\mathcal{E}$ and the Groverian measure G are plotted in figure 3. As expected, all entanglement meaures reduce to zero at

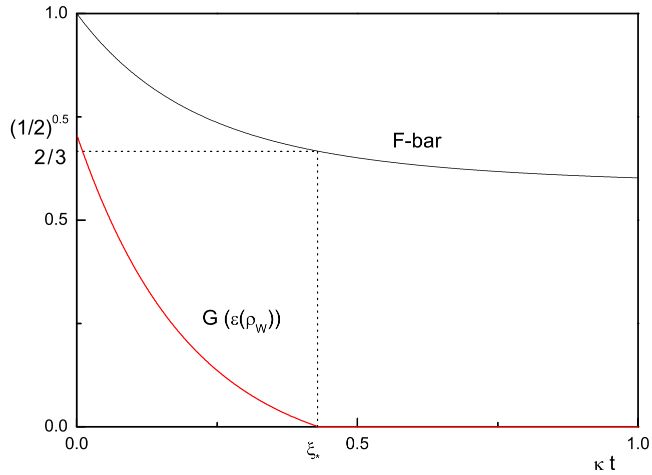

Figure 4. The κt-dependence of the average fidelity $\bar{F}$ and the Groverian measure G for 3-qubit mixed state $\varepsilon(\rho_W)$. The optimal ensemble for $\varepsilon(\rho_W)$ should make G zero when $\kappa t \geqslant \xi_*$. This may give valuable information for the construction of the optimal ensemble for higher-qubit states.

$\kappa t \geqslant \nu_*$. Thus our conjecture described in section 1 is perfectly correct. This fact indicates that the entanglement of the quantum channel is a genuine physical resource in the teleportation process. Also our conjecture may offer valuable clues to the optimal decomposition in the higher-qubit mixed states. This will be discussed briefly in the following section.

5. Conclusion

In this paper we have examined the connection between the mixed-state entanglement and the average fidelity $\bar{F}$ using the usual EPR-state teleportation via noises. As we have shown, the mixed-state entanglement becomes zero when $\bar{F} \leqslant 2/3$, which indicates that the entanglement of quantum channel is a genuine resource for teleportation.

It is generally a non-trivial task to compute the entanglement of n-qubit mixed states when $n \geqslant 3$. As far as we know, in addition, there is no way to find an optimal ensemble of pure states when $n \geqslant 3$. Also we cannot define the concurrence because there is no 'magic'-like basis in the higher-qubit system. However, the result of our paper may provide a valuable information on the entanglement of higher-qubit mixed states. For example, let us consider the 3-qubit mixed state

$$\varepsilon(\rho_W) = \frac{1}{16}\begin{pmatrix} 2\alpha_2 & 0 & 0 & \sqrt{2}\alpha_2 & 0 & \sqrt{2}\alpha_2 & \alpha_2 & 0 \\ 0 & 2\alpha_1 & \sqrt{2}\alpha_1 & 0 & \sqrt{2}\alpha_1 & 0 & 0 & \alpha_3 \\ 0 & \sqrt{2}\alpha_1 & 2\beta_+ & 0 & \alpha_1 & 0 & 0 & \sqrt{2}\alpha_3 \\ \sqrt{2}\alpha_2 & 0 & 0 & 2\beta_- & 0 & \alpha_4 & \sqrt{2}\alpha_4 & 0 \\ 0 & \sqrt{2}\alpha_1 & \alpha_1 & 0 & 2\beta_+ & 0 & 0 & \sqrt{2}\alpha_3 \\ \sqrt{2}\alpha_2 & 0 & 0 & \alpha_4 & 0 & 2\beta_- & \sqrt{2}\alpha_4 & 0 \\ \alpha_2 & 0 & 0 & \sqrt{2}\alpha_4 & 0 & \sqrt{2}\alpha_4 & 2\alpha_4 & 0 \\ 0 & \alpha_3 & \sqrt{2}\alpha_3 & 0 & \sqrt{2}\alpha_3 & 0 & 0 & 2\alpha_3 \end{pmatrix} \tag{5.1}$$

where

$$\begin{aligned}\alpha_1 &= 1 + e^{-2\kappa t} + e^{-4\kappa t} + e^{-6\kappa t}\\ \alpha_2 &= 1 + e^{-2\kappa t} - e^{-4\kappa t} - e^{-6\kappa t}\\ \alpha_3 &= 1 - e^{-2\kappa t} - e^{-4\kappa t} + e^{-6\kappa t}\\ \alpha_4 &= 1 - e^{-2\kappa t} + e^{-4\kappa t} - e^{-6\kappa t}\\ \beta_\pm &= 1 \pm e^{-6\kappa t}.\end{aligned} \tag{5.2}$$

This mixed state is constructed when the quantum teleportation is performed with the W-state

$$|\psi_W\rangle = \tfrac{1}{2}(|100\rangle + |010\rangle + \sqrt{2}|001\rangle) \tag{5.3}$$

if $(L_{2,x}, L_{3,x}, L_{4,x})$ noise is introduced [26]. It has been shown in [26] that its average fidelity between Alice and Bob is

$$\bar{F} = \tfrac{1}{24}(14 + 3\,e^{-2\kappa t} + 2\,e^{-4\kappa t} + 5\,e^{-6\kappa t}). \tag{5.4}$$

Thus $\bar{F}$ decreases from 1 to 7/12 as κt goes from 0 to ∞. From this fact we can conjecture that the Groverian measure (2.2) for $\varepsilon(\rho_W)$ decreases from $1/\sqrt{2}$ to 0 when κt goes from 0 to $\xi_* = 0.431\,041$ if we find the optimal ensemble of pure states for this mixed state. This conjecture is described in figure 4. This information may give valuable clues to the construction of the optimal ensemble of pure states in the 3- or higher-qubit system.

Another point we would like to note is on the second definition of the Groverian measure $\tilde{G}(\rho)$ defined in equation (2.3). Since it is not defined by the convex roof construction due to its operational meaning, we cannot use the usual optimal ensemble technique to compute it. Since, furthermore, it is expressed as equation (2.4) via Uhlmann's theorem, we should know how to compute the Groverian measure of n-qubit pure states with $n \geqslant 4$. Even if we assume that we have the formula for the n-qubit pure-state Groverian measure, it is also highly non-trivial to take a maximization over all possible purification. Since, however, it is a genuine entanglement measure for mixed states, it should satisfy our conjecture. It may shed light on the development of the computational technique for $\tilde{G}(\rho)$ in the future.

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